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TECHNICAL REPORT RD-81-1

# Z-TRANSFORM TECHNIQUES FOR IMPROVED REAL-TIME DIGITAL SIMULATION OF CONTINUOUS SYSTEMS

## RUNGE-KUTTA CONVOLUTIONS ADJUSTED FOR UNIT STEP RESPONSE VIA POLE-RESIDUES

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1 October 1980



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SUBJECT: Errata for Technical Report RD-81-1, subject: Z-TRANSFORM TECHNIQUES FOR IMPROVED REAL-TIME DIGITAL SIMULATION OF CONTINUOUS SYSTEMS: RUNGE-KUTTA CONVOLUTIONS ADJUSTED FOR UNIT STEP RESPONSE VIA POLE-RESIDUES, dated 1 Oct 80

TO: Recipients of Subject Report

The following changes should be made in subject report, attached:

Equation (51), page 13, should read:

$$\frac{N}{(e^{-aT} - e^{-bT})/(b - a)} = \frac{-(b - a)T}{2\alpha} \; .....$$

Equation (55), page 14, should read:

$$\frac{-[(\sigma - \Sigma) + i(\omega - \Omega)]T \left(e^{-\alpha[(\sigma - \Sigma) + i(\omega - \Omega)]T} + (2\alpha - 1)\right)}{2\alpha\left(e^{-[(\sigma - \Sigma) + i(\omega - \Omega)]T} - 1\right)} .$$

Equation (56), page 14, should read:

$$Re = \frac{-(\sigma - \Sigma)T \; A + (\omega - \Omega)T \; B}{2\,\alpha\,D} ,$$

Equation (57), page 14, should read:

$$Im = \frac{-i(\omega - \Omega)T \; A - i(\sigma - \Sigma)T \; B}{2\,\alpha\,D} ,$$

Equation (59), page 14, should read:

$$A = \ldots$$

$$+ (2\alpha - 1)(e^{-(\sigma - \Sigma)T} \cos[(\omega - \Omega)T] - 1)$$

$$= \quad \ldots$$

1 Incl
as

Linda Yancey
for JOHN W. CHAMBERS
Ch, Tech Info Div
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| 20. ABSTRACT (Continue on reverse side if necessary and identify by block number)<br>An approximation for Z[G(s)F(s)] is developed using the second order Runge-Kutta integrator family. The residue for exact unit step response of a single real pole filter is then developed. The error for a complex exponential into a complex pole is also presented. | | |

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# I. INTRODUCTION

In recent decades z-transform techniques have found use in the digital simulation of continuous linear systems. One of these techniques[1] is compared with a classical method[2] and possible methods for improvement of the technique are suggested. The emphasis will be in making the time step, T, as large as possible for speed in real-time simulation or economy in Monte Carlo studies. Another possibility would be improvements in the fidelity of the simulation. The approach would also find application in microprocessor and microcomputer programming.

Since the steady state response of a system is of interest, the response of the plant to the Heaviside unit step,

$$u(t) = \begin{cases} 0, & t < 0 \\ 1, & t \geq 0 \end{cases} \quad , \tag{1}$$

will be analyzed. Other inputs considered are the damped exponential and sine-wave.

The single pole filter will serve as a test case since it is the basic transfer function. The technique implies the use of the Jordan canonical form.[3]

Before proceeding further, some comments on the conventions used in this report are warranted (Figure 1). From the "sifting" property of the Dirac delta distribution

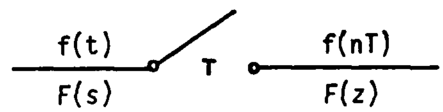



Figure 1. Time, Sample Data, Frequency and Sequency "Domains."

$$f(nT) = \int_{-\infty}^{\infty} f(t)\ \delta(nT - t)\ dt \ . \tag{2}$$

The Laplace transform is defined to be

$$F(s) = \int_{-\infty}^{\infty} f(t)\ u(t)\ e^{-st}\ dt \tag{3}$$

and the z-transform, a "discrete" Laplace transform,

$$F(z) = \int_{-\infty}^{\infty} \sum_{n=-\infty}^{\infty} f(t)\ \delta(nT - t)\ u(t)\ e^{-st}\ dt \ , \tag{4}$$

which readily reduces to

$$F(z) = \sum_{n=-\infty}^{\infty} f(nT)\ u(nT)\ e^{-snT} \ . \tag{5}$$

For

$$z = e^{-sT} ,$$

Equation (5) becomes[3]

$$F(z) = \sum_{n=0}^{\infty} f(nT)\ z^{n} . \tag{6}$$

The modified z-transform is

$$F(z,\alpha) = \sum_{n=0}^{\infty} F(nT + \alpha T)\ z^{n} , \tag{7}$$

and finally

$$L\left[f^{(n)}(t)\right] = s^{n}F(s) - \sum_{\ell=0}^{n-1} s^{n-\ell-1}\ f^{(\ell)}(0) , \tag{8}$$

which is used to incorporate non-zero initial conditions.

Since the z-transform is developed from the known time domain solution of a linear differential equation, the recurrences developed for a given input and plant are exact for any size time step, T.[4]

Transforming the linear differential equation for the single pole filter with a unit step input,

$$\dot{y}(t) + a\ y(t) = u(t) , \tag{9}$$

to the Laplace domain, using Equation (8) yields

$$\left[s\ Y(s) + y(0)\right] + a\ Y(s) = \frac{1}{s} . \tag{10}$$

Collecting terms yields

$$Y(s) = \frac{y(0)}{s + a} + \frac{1}{s(s + a)} . \tag{11}$$

Taking the z-transform of Equation (11),

$$Y(z) = y(0)\ Z\left[\frac{1}{s + a}\right] + Z\left[\frac{1}{s(s + a)}\right] . \tag{12}$$

Since the initial condition is a constant in the Laplace domain (an impulse in the time domain), the Raggazzini-Zadeh identity applies.[1,5]

The z-transform for a unit step into a single pole filter is[5]

$$Z\left[\frac{1}{s(s + a)}\right] = \frac{(1 - e^{-aT})z}{a(1 - z)(1 - e^{-aT}z)} \ . \tag{13}$$

Substituting Equation (13) into Equation (12) yields

$$Y(z) = \frac{y(0) + (1 - e^{-aT})z/a(1 - z)}{1 - e^{-aT}z} \ . \tag{14}$$

Noting that

$$\frac{1}{1 - z} = \sum_{n=0}^{\infty} z^n \ , \tag{15}$$

and equating coefficients of like powers of "z" in Equation (14),

$$y(0) = y(0) \ , \tag{16a}$$

$$y(nT) = e^{-aT} \ y(nT - T) + (1 - e^{-aT})/a, \ n > 0 \ . \tag{16b}$$

With a little mathematical induction, from

$$y(T) = e^{-aT} \ y(0) + (1 - e^{-aT})/a \tag{17}$$

$$y(2T) = e^{-aT} \ y(T) + (1 - e^{-aT})/a \tag{18a}$$

$$= e^{-aT} \left[e^{-aT} y(0) + (1 - e^{-aT})/a\right] + (1 - e^{-aT})/a \tag{18b}$$

$$= e^{-2aT} y(0) + (1 - e^{-2aT})/a \tag{18c}$$

it follows that

$$y(nT) = e^{-naT} y(0) + (1 - e^{-naT})/a \ , \tag{19}$$

the known solution to the differential equation, Equation (9).

## II. CLASSICAL APPROACH

Now consider a second order Runge-Kutta integration for the solution of Equation (2); actually all second order Runge-Kutta integrators[2] are considered,

$$y(n) \approx y(n - 1) + T\left[\left(1 - \frac{1}{2\alpha}\right) \dot{y}(n - 1) + \left(\frac{1}{2\alpha}\right) \dot{y}(n - (1 - \alpha))\right] , \tag{20a}$$

$$\dot{y}(n - 1) = 1 - a \ \ y(n - 1) \ , \tag{20b}$$

$$\dot{y}[n - (1 - \alpha)] \simeq 1 - a[y(n - 1) + \alpha T\, \dot{y}(n - 1)] \; . \tag{20c}$$

Substituting and collecting terms,

$$y(n) \simeq (1 - aT + (aT)^2/2)\; y(n - 1) + [1 - (1 - aT + (aT)^2/2)]/a \; ; \tag{21}$$

the $\alpha$'s cancel out!

Note that

$$e^{-aT} \simeq 1 - aT + (aT)^2/2 \tag{22}$$

is the (2/0) Padé approximation for the exponential.[6] This is not unexpected since the Runge-Kutta integrators were developed by truncating the Taylor series, and the (N/0) Padé approximations for the exponential are just the truncated Taylor series.

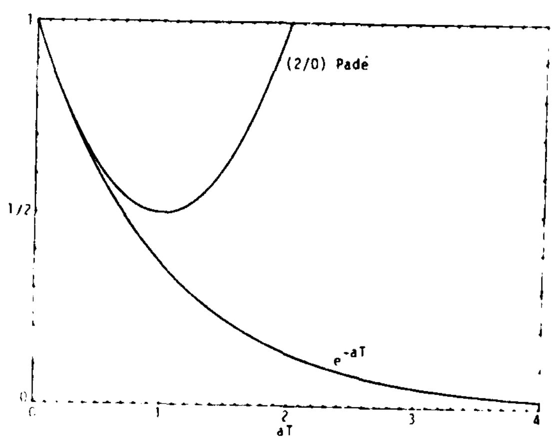
In the sequency domain, Equation (14) would be

$$Y(z) \simeq \frac{y(0) + [1 - (1 - aT + (aT)^2/2)]z/a(1 - z)}{1 - (1 - aT + (aT)^2/2)z} \; . \tag{23}$$

To develop a recurrence for the error[7] of the Runge-Kutta integrator, subtract Equation (4) from Equation (23):

$$\varepsilon(z) = \frac{y(0) + [1 - (1 - aT + (aT)^2/2)]z/a(1 - z)}{1 - (1 - aT + (aT)^2/2)z} - \frac{y(0) + (1 - e^{-aT})z/a(1 - z)}{1 - e^{-aT}z} \; . \tag{24}$$

Without proceeding further, it is readily apparent that the initial condition does not cancel out.




In Figure 2, the (2/0) Padé approximation is plotted with the exact function. The approximation not only becomes inaccurate when aT is too large, but exceeds 1 for aT greater than 2.

Figure 2. (2/0) Padé for $e^{-aT}$.

## III. A Z-TRANSFORM APPROACH

Of course, the z-transform is exact when the input and plant are both given a priori. In a digital simulation, the input is a sequence of values and not a known function, that is, it is not generally known a priori.

In this case, approximations for the z-transform of a product in the Laplace domain are required. The problem is to approximate the convolution integral[1,5] in

$$Z\left[G(s)\ F(s)\right] = \sum_{n=-\infty}^{\infty} z^n\ u(nT) \int_{-\infty}^{\infty} g(t)\ u(t)\ f(nT - t)\ u(nT - t)dt \tag{25a}$$

$$= \sum_{n=-\infty}^{\infty} z^n\ u(nT) \sum_{k=-\infty}^{\infty} \int_{kT}^{kT+T} g(t)\ u(t)\ f(nT - t)\ u(nT - t)dt\ . \tag{25b}$$

Those approximations previously developed[1,5,8] are shown in Table 1.

TABLE 1. RUNGE-KUTTA CONVOLUTIONS

| $Z\left[G(s)\ F(s)\right]$ |
|---|
| $z\ T\ G\left(z,\frac{1}{2}\right)\ F\left(z,\frac{1}{2}\right)$ |
| $\frac{z\ T}{2}\left[G(z)\ F(z,1) + G(z,1)\ F(z)\right]$ |
| $\frac{z\ T}{6}\left[G(z)\ F(z,1) + 4\ G\left(z,\frac{1}{2}\right)\ F\left(z,\frac{1}{2}\right) + G(z,1)\ F(z)\right]$ |
| $\frac{z\ T}{8}\left[G(z)\ f(z,1) + 3\ G\left(z,\frac{1}{3}\right)\ F\left(z,\frac{2}{3}\right) + 3\ G\left(z,\frac{2}{3}\right)\ F\left(z,\frac{1}{3}\right) + G(z,1)\ F(z)\right]$ |

The second order Runge-Kutta integrator family [see Equation (20a)] leads to

$$\int_{kT}^{kT+T} g(t)\ \varepsilon(nT - t)dt \simeq T\left(1 - \frac{1}{2\alpha}\right) g(kT)\ f(nT - kT)$$

$$+ T\left(\frac{1}{2\alpha}\right) g(kT + T)\ f(nT - (k + \alpha)T)\ . \tag{26}$$

Substituting Equation (26) into Equation (25b) would lead to the desired results after some manipulation of the series.[5] A shortcut is available; note that on the right-hand side of Equation (26) the first term leads to Eular Convolution and the second to Mean Value Convolution. With this insight, the

results may be written down by inspection as was done before for some higher order Runge-Kutta convolutions,[5]

$$Z\left[G(s)\ F(s)\right] \simeq T\left(1 - \frac{1}{2\alpha}\right) G(z)\left[F(z) - f(0)\right]$$

$$+ T\left(\frac{1}{2\alpha}\right) G(z,\alpha)\left[F(z,-\alpha) - f(-\alpha T)\right] \ , \tag{27}$$

R-K(2,$\alpha$) Convolution. Eular Convolution and Mean Value Convolution are just special cases of R-K(2,$\alpha$) Convolution (see Table 2).

TABLE 2. SPECIAL CASES OF R-K(2,$\alpha$)C

| Convolution | $\alpha$ |
|---|---|
| Eular | 0 |
| Mean Value for 1/2 | 1/2 |
| Trapezoidal | 1 |

For a single real pole filter,

$$F(s) = \frac{1}{s + a} \ , \tag{28}$$

and any input, G(s), the approximation using R-K(2,$\alpha$)C is

$$Z\left[\frac{G(s)}{s + a}\right] \simeq T\left(1 - \frac{1}{2\alpha}\right) G(z)\left[\frac{1}{1 - e^{-aT}z} - 1\right]$$

$$+ T\left(\frac{1}{2\alpha}\right) G(z,\alpha)\left[\frac{e^{\alpha aT}}{1 - e^{-aT}z} - e^{\alpha aT}\right] \tag{29a}$$

$$\simeq \frac{T\left(1 - \frac{1}{2\alpha}\right) e^{-aT}z\ G(z) + T\left(\frac{1}{2\alpha}\right) e^{-(1-\alpha)aT}z\ G(z,\alpha)}{1 - e^{-aT}z} \ . \tag{29b}$$

The initial condition would be incorporated exactly as in Equations (9) - (16a).

Equating coefficients of like powers of "z," one has the recurrence

$$y(n) \simeq e^{-aT}\ y(n - 1) + T\left[\left(1 - \frac{1}{2\alpha}\right) e^{-aT}\ g(n - 1)\right.$$

$$\left. + \left(\frac{1}{2\alpha}\right) e^{-(1-\alpha)aT}\ g(n - (1 - \alpha))\right] , \ n > 0 \ . \tag{30}$$

For a Heaviside unit step input, $G(s) = \frac{1}{s}$ ,

$$Z\left[\left(\frac{1}{s}\right)\left(\frac{1}{s + a}\right)\right] \simeq \frac{zT\left[\left(1 - \frac{1}{2\alpha}\right)e^{-aT} + \left(\frac{1}{2\alpha}\right)e^{-(1-\alpha)aT}\right]}{(1 - z)(1 - e^{-aT}z)} . \tag{31}$$

While the denominator of Equation (31) is exact, in the numerator

$$e^{-aT} \simeq 1 - aT\left[\left(1 - \frac{1}{2\alpha}\right) e^{-aT} + \left(\frac{1}{2\alpha}\right) e^{-(1-\alpha)aT}\right] . \tag{32}$$

As aT becomes very large, the right-hand side of Equation (32) approaches unity. Plots of Equation (32) are shown in Figure 3.

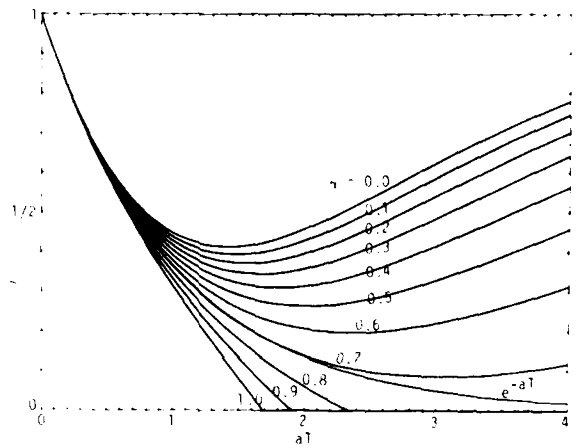



Figure 3. Plots of Equation (32).

A single integrator will integrate a Heaviside unit step exactly for any size time step,[5] but the recurrence for a single real pole filter will not. For what input, if any, is the recurrence exact?

Two exact z-transforms of interest are[5]

$$Z\left[\frac{1}{(s + b)(s + a)}\right] = \frac{(e^{-aT} - e^{-bT})z}{(b - a)(1 - e^{-bT}z)(1 - e^{-aT}z)} \tag{33a}$$

and

$$Z\left[\frac{1}{(s + a)^2}\right] = \frac{T\, e^{-aT}z}{(1 - e^{-aT}z)^2} . \tag{33b}$$

For R-K(2,α)C, Equation (27), one has

$$Z\left[\left(\frac{1}{s+b}\right)\left(\frac{1}{s+a}\right)\right] \approx T\left(1 - \frac{1}{2\alpha}\right)\left(\frac{1}{1 - e^{-bT}z}\right)\left[\frac{1}{1 - e^{-aT}z} - 1\right]$$

$$+ T\left(\frac{1}{2\alpha}\right)\left(\frac{e^{-\alpha bT}}{1 - e^{-bT}z}\right)\left[\frac{e^{\alpha aT}}{1 - e^{-aT}z} - e^{\alpha aT}\right] \tag{34a}$$

$$\approx \frac{T\,z\left[\left(1 - \frac{1}{2\alpha}\right)e^{-aT} + \left(\frac{1}{2\alpha}\right)e^{-\alpha bT}e^{-(1-\alpha)aT}\right]}{(1 - e^{-bT}z)(1 - e^{-aT}z)} . \tag{34b}$$

No choice of α leads to Equation (33a). For a = b, Equation (34b) becomes

$$Z\left[\left(\frac{1}{s+a}\right)\left(\frac{1}{s+a}\right)\right] \approx \frac{T\,e^{-aT}z}{(1 - e^{-aT}z)(1 - e^{-aT}z)} \tag{33b}$$

which is exact for any αT! This implies that the recurrence for a single real pole filter can filter a decaying exponential exactly for any size time step, T, where they have identical time constants (a = b).

The unit step into a single integrator is just a special case (a = b = 0). From this point of view, the single integrator is just a single pole filter at the origin; the recurrence for the single pole filter reduces to the classical integrator(s) for a = 0.

## IV. ADJUSTMENT OF UNIT STEP RESPONSE VIA RESIDUES

It is generally more desirable to have the recurrence for a single real pole filter respond exactly to the Heaviside unit step. To determine how this might be accomplished, it is necessary to first determine the error for a unit step input into a single real pole filter. To develop a recurrence for the error,[7] subtract the exact z-transform from the convolution approximation; Equation (31) minus Equation (13),

$$\varepsilon(z) = \frac{\left[N - (1 - e^{-aT})/a\right]z}{(1 - z)(1 - e^{-aT}z)} , \tag{35}$$

where

$$N = T\left[\left(1 - \frac{1}{2\alpha}\right)e^{-aT} + \left(\frac{1}{2\alpha}\right)e^{-\alpha bT}\,e^{-(1-\alpha)aT}\right] . \tag{36}$$

Since the initial condition would be incorporated exactly,[1,5] it would vanish.

Equating coefficients of like powers of z,

$$\varepsilon(0) = 0 \ , \tag{37a}$$

$$\varepsilon(nT) = e^{-aT}\ \varepsilon(nT - T) + \left[N - (1 - e^{-aT})/a\right], \quad n > 0 \ . \tag{37b}$$

Writing a few terms

$$\varepsilon(T) = N - (1 - e^{-aT})/a$$

$$\varepsilon(2T) = (1 + e^{-aT})\left[N - (1 - e^{-aT})/a\right] \ , \tag{38a}$$

it becomes apparent that

$$\varepsilon(nT) = \left(\sum_{k=0}^{n-1} e^{-kaT}\right)\left[N - (1 - e^{-aT})/a\right] \ . \tag{38c}$$

Factoring yields

$$\varepsilon(nT) = \frac{1 - e^{-aT}}{a}\left(\sum_{k=0}^{n-1} e^{-kaT}\right)\left[\frac{N}{(1 - e^{-aT})/a} - 1\right] \ . \tag{39}$$

To remove the error, introduce a residue into Equation (29a),

$$Z\left[G(s)\left(\frac{Res}{s + a}\right)\right] \simeq Res\ T\left[\left(1 - \frac{1}{2\alpha}\right) G(z)\ e^{-aT}\right.$$

$$\left. + \left(\frac{1}{2\alpha}\right) e^{-(1-\alpha)aT} z\ G(z,\alpha)\right]/(1 - e^{-aT} z) \ . \tag{40}$$

Equation (39) would then become

$$\varepsilon(nT) = \frac{1 - e^{-aT}}{a}\left(\sum_{k=0}^{n-1} e^{-kaT}\right)\left[\frac{Res\ N}{(1 - e^{-aT})/a} - 1\right] \ . \tag{41}$$

Setting the quantity in the brackets equal zero gives

$$Res = \frac{(1 - e^{-aT})/a}{N} \ . \tag{42}$$

For this residue, Equation (42), the recurrence

$$y(n) \approx e^{-aT} y(n-1) + \text{Res}\, T\left[\left(1 - \frac{1}{2\alpha}\right) e^{-aT} g(n-1)\right.$$

$$\left. + \left(\frac{1}{2\alpha}\right) e^{-(1-\alpha)aT} g(n-(1-\alpha))\right], \quad n > 0 \tag{43}$$

would be exact for a unit step input for any size time step, T!

For $\alpha = 1$, that is, Trapezoidal Convolutions[8]

$$N = \frac{T}{2}(1 + e^{-aT}) \tag{44}$$

and

$$\text{Res} = \frac{(1 - e^{-aT})/a}{\frac{T}{2}(1 + e^{-aT})}, \tag{45a}$$

$$\text{Res} = \frac{2}{aT} \tanh\left(\frac{aT}{2}\right). \tag{45b}$$

In this case, the recurrence, Equation (43), would be

$$y(n) \approx e^{-aT} y(n-1) + \frac{\tanh\left(\frac{aT}{2}\right)}{a}\left[e^{-aT} g(n-1) + g(n)\right], \quad n > 0, \tag{46}$$

instead of

$$y(n) \approx e^{-aT} y(n-1) + \frac{T}{2}\left[e^{-aT} g(n-1) + g(n)\right], \quad n > 0. \tag{47}$$

## V. ACCURACY ANALYSIS

Instead of the error for a unit step input, consider the error for an exponential input into a single pole filter. Equations (35b) minus Equation (33) yields

$$\epsilon(z) = \frac{z\,N - z(e^{-aT} - e^{-bT})/(b-a)}{(1 - e^{-bT}z)(1 - e^{-aT}z)}. \tag{48}$$

Since the initial condition is incorporated exactly in either case, it vanishes.

Dividing by the exact z-transform, Equation (33a),

$$\frac{\varepsilon(z)}{Z\left[\frac{1}{(s+b)(s+a)}\right]} = \frac{N}{(e^{-aT} - e^{-bT})/(b-a)} - 1 \ . \qquad (49)$$

The rationale for Equation (49) is as follows: Since the initial condition is incorporated exactly, any error due to the approximations is in the lower channel of Figure 4. Dividing by Y(z) would introduce the initial condition into the expression for the (percent) error.

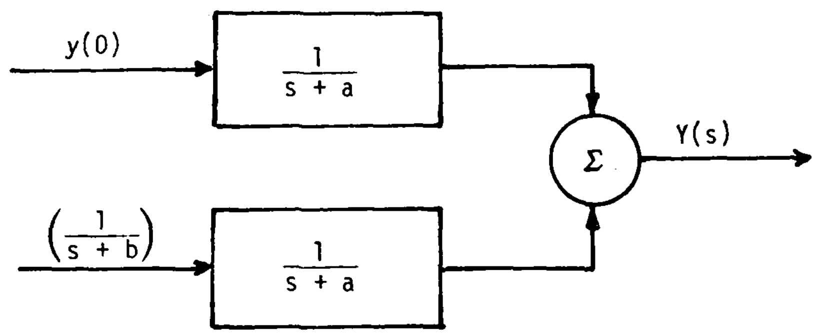



Figure 4. Single Real Pole Filter.

Only the error in the lower channel is being considered and not the error in Y(z). Of course, if the initial conditions are zero, there is no difficulty because there is no difference. In this case,

$$Y(z) = Z\left[\frac{1}{(s+b)(s+a)}\right] \ . \qquad (50)$$

There is ample precedence for this approach in transfer function analysis. To have the ratio of output to input all the initial conditions must be zero.

Factoring out $e^{aT}$,

$$\frac{N}{(e^{-aT} - e^{-bT})/(b-a)} = \frac{(b-a)T}{2\alpha}\left[\frac{e^{-\alpha(b-a)T} + (2\alpha - 1)}{e^{-(b-a)T} - 1}\right] \ . \qquad (51)$$

In the preceding it has been tacitly implied that "a" and "b" are both real but there is nothing to prevent them from being imaginary or complex. In fact, this approach was motivated by an analysis of a sine wave input ($b = \omega$) into single integrators ($a = 0$).[3,9,10] Actually, the input considered was

$$e^{i\omega T} = \cos \omega T + i \ \sin \omega T \ , \qquad (52)$$

which yields both amplitude and phase response.[5]

For

$$b = \sigma + i\omega \tag{52}$$

and

$$a = \Sigma + i\Omega \tag{53}$$

the right-hand side of Equation (51) becomes

$$\frac{\left[(\sigma - \Sigma) + i(\omega - \Omega)\right]T \left(e^{-\alpha[(\sigma-\Sigma)+i(\omega-\Omega)]T} + (2\alpha - 1)\right)}{2\alpha\left(e^{-[(\sigma-\Sigma)+i(\omega-\Omega)]T} - 1\right)} \ . \tag{55}$$

Substituting Eular's Rule, Equation (52), multiplying top and bottom by the complex conjugate of the denominator and collecting term: for the real part

$$\mathrm{Re} = \frac{(\sigma - \Sigma)T\ A - (\omega - \Omega)T\ B}{2\ \alpha\ D} \ , \tag{56}$$

and for the imaginary part

$$\mathrm{Im} = \frac{i(\omega - \Omega)T\ A + i(\sigma - \Sigma)T\ B}{2\ \alpha\ D} \ , \tag{57}$$

where

$$D = e^{-2(\sigma-\Sigma)T} - 2e^{-(\sigma-\Sigma)T}\cos\left[(\omega - \Omega)T\right] + 1 \tag{58}$$

$$\begin{aligned} A = {} & e^{-(1+\alpha)(\sigma-\Sigma)T}\cos\left[(1 - \alpha)(\omega - \Omega)T\right] \\ & + (2\alpha - 1)\ e^{-(\sigma-\Sigma)T}\cos\left[(\omega-\Omega)T\right] - 1 \\ & - e^{-\alpha(\sigma-\Sigma)T}\cos\left[\alpha(\omega - \Omega)T\right] \end{aligned} \tag{59}$$

$$\begin{aligned} B = {} & e^{-(1+\alpha)(\sigma-\Sigma)T}\sin\left[(1 - \alpha)(\omega - \Omega)T\right] \\ & + (2\alpha - 1)\ e^{-(\sigma-\Sigma)T}\sin\left[(\omega - \Omega)T\right] \\ & + e^{-\alpha(\sigma-\Sigma)T}\sin\left[\alpha(\omega - \Omega)T\right] \ . \end{aligned} \tag{60}$$

Of course, to compute the amplitude ratio:

$$(\mathrm{Re}^2 + \mathrm{Im}^2)^{1/2} \tag{61}$$

while the phase error is

$$\arctan\ (\mathrm{Im}/\mathrm{Re}) \ . \tag{62}$$

Note that if either $\omega = \Omega$ or $\sigma = \Sigma$, there is no phase error when $\alpha = 1$.

The ranges of values in Figures 5-18 require some explanation. The imaginary part, $(\omega - \Omega)T$, ranges from 0 to $\pm\pi(\pm 180^{o})$. The upper value is the Shannon sampling limit of two samples per cycle. The curves for $\pm\pi/2(\pm 90^{o})$, that is, twice the Shannon limit or four samples per cycle, are a more realistic boundary on performance. The real part, $(\sigma - \Sigma)T$, ranges from -4 to +4. A realistic working limit appears to be ±2. If the realistic limit had been used on the plots, it would not have been clear what happens when these limits are exceeded.

## VI. RESULTS AND RECOMMENDATIONS

A "convolution" approximation based upon the Runge-Kutta second order integrators was derived, Equation (27). For a unit step into a single real pole filter the approximation, R-K(2,$\alpha$)C, proved superior to the Runge-Kutta integrators, R-K(2,$\alpha$)I; compare Figure 2 with Figure 3.

It was then shown that when the input and the plant are identical single poles, the R-K(2,$\alpha$)C approximation is exact for any $\alpha$, Equation (34). This rather interesting observation was then used to develop a residue, Equation (42), which makes the approximation exact for a unit step input to a single real pole filter.

Then an analysis was made of the amplitude and phase response of a complex exponential into a complex pole for R-K(2,$\alpha$)C. A value of $\alpha = 2/3$ offers excellent amplitude (Figure 13) and phase (Figure 14) response over a wide range of complex values.

Since the amplitude (Figure 5) and phase (Figure 6) response for R-K(2,1)C, Trapezoidal Convolution, is not so well behaved, R-K(2,2/3)C should be considered for real time applications when a large time step, T, is required. In a "classical" simulation the Runge-Kutta second order integration with $\alpha = 2/3$ is recommended.

There may be some difficulties when the problem is implicit. In this case Eular Convolution ($\alpha = 0$) is required to "predict" the advanced value for the recurrence. Van der Pol's equation would serve as a test bed for the ability of the technique to handle implicit, nonlinear problems, and such a study has been proposed as a follow-on to this effort.

This technique requires further development and refinement, and is in no way complete at this time, but results to date are very promising.

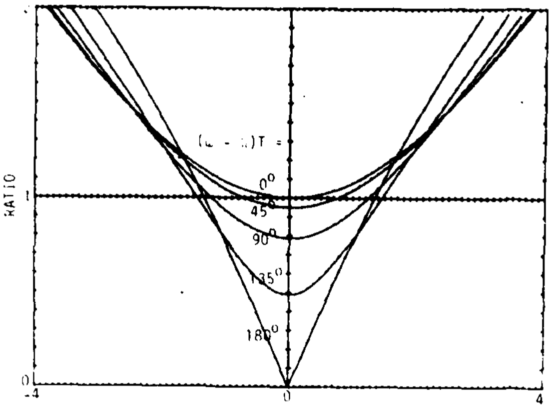



Figure 5. Amplitude Ratio for R-K(2,1)C.

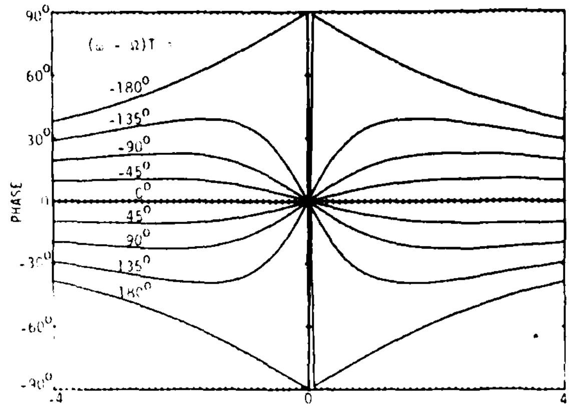



Figure 6. Phase Error for R-K(2,1)C.

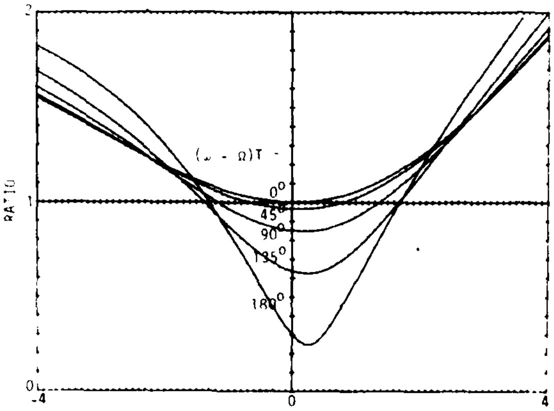



Figure 7. Amplitude Ratio for R-K(2,0.9)C.

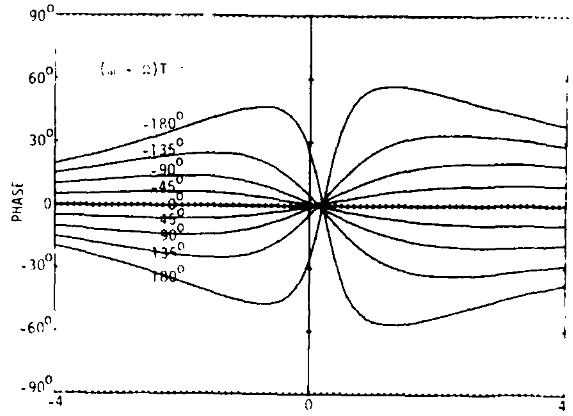



Figure 8. Phase Error for R-K(2,0.9)C.

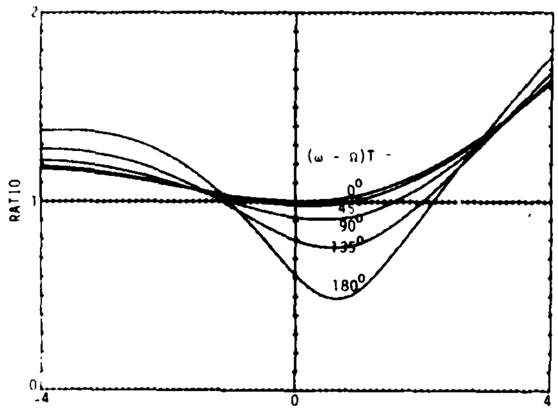



Figure 9. Amplitude Ratio for R-K(2,0.8)C.

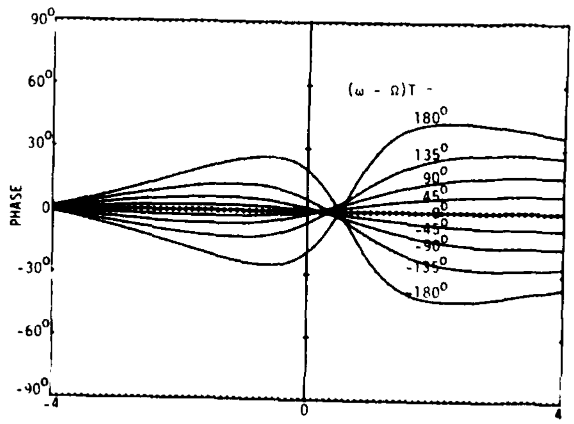



Figure 10. Phase Error for R-K(2,0.8)C.

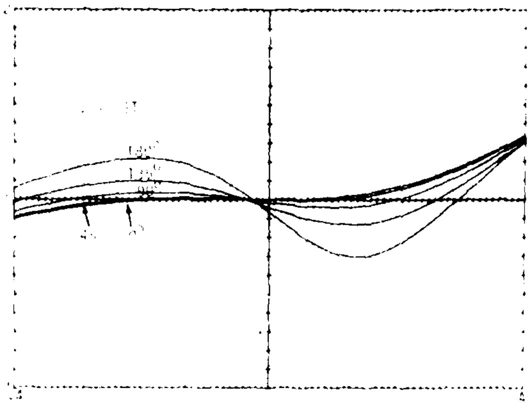



Figure 11. Amplitude Ratio for R-K(2,0.7)C.

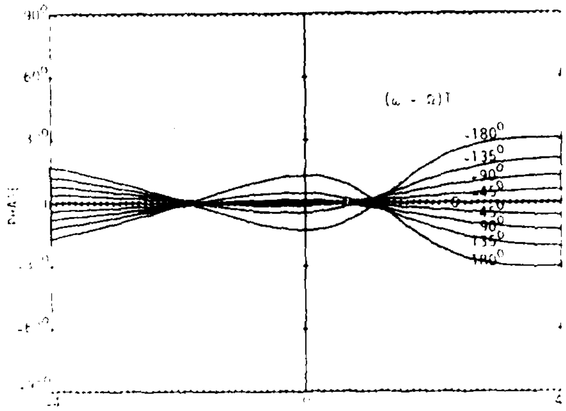



Figure 12. Phase Error for R-K(2,0.7)C.

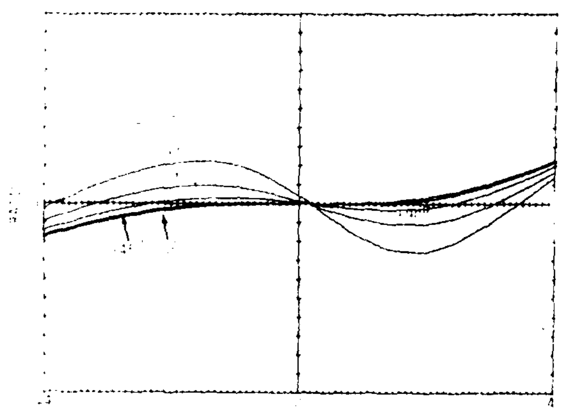



Figure 13. Amplitude Ratio for R-K(2, 2/3)C.

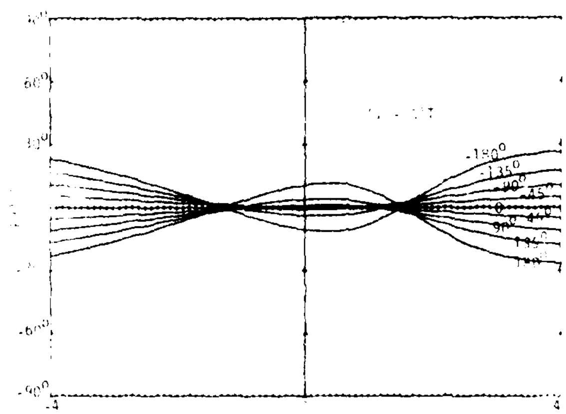



Figure 14. Phase Error for R-K(2, 2/3)C.

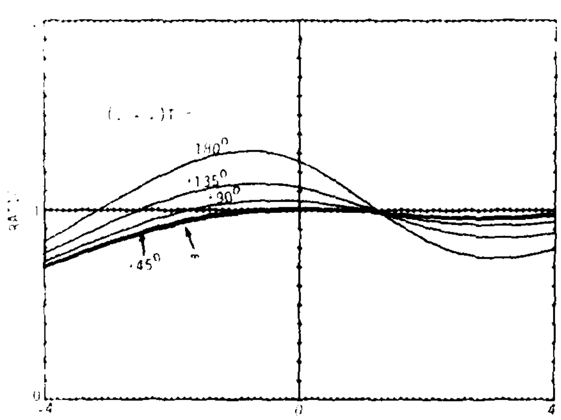



Figure 15. Amplitude Ratio for R-K(2,0.6)C.

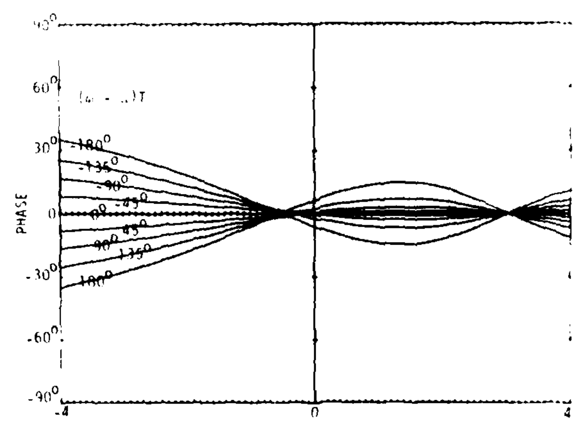



Figure 16. Phase Error for R-K(2,0.6)C.

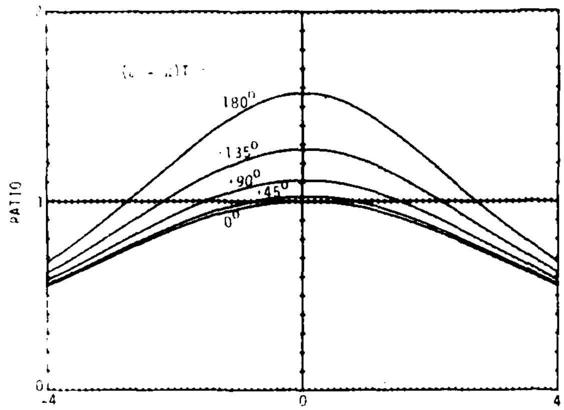



Figure 17. Amplitude Ratio for R-K(2,0.5)C.

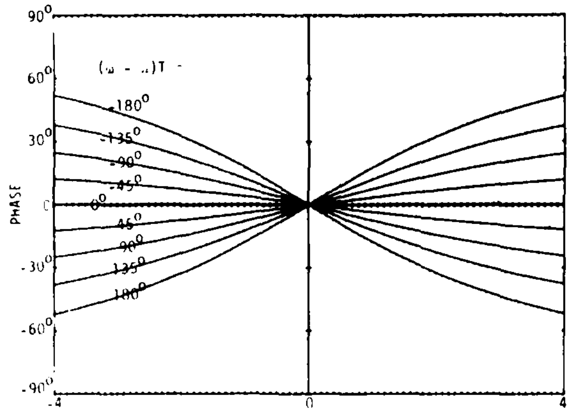



Figure 18. Phase Error for R-K(2,0.5)C.

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